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Discovery of Closed Orbits of Dynamical Systems with the Use of Computers

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In this paper we derive a general criterion which can be used for the discovery with the use of a computer of closed orbits of systems of ordinary differential equations. We apply this criterion to the Lorenz model and show rigorously the existence of a closed orbit for the case under consideration. In a subsequent paper we shall show how the stable manifold of this orbit determines the boundary of the stochastic attractor.

KEY WORDS: Poincaré mapping; linear system of equations in variations; closed orbit; attractor.

1. INTRODUCTION

One of the most striking recent results in the theory of dynamical systems is the discovery of many examples of dynamical systems described by rather simple systems of ordinary differential equations where numerical investigations show the presence of stochastic behavior (Lorenz model, Henon attractor, etc.).[1–3] There is no doubt that in many cases the rigorous treatment of such systems will be based upon information obtained with the help of computers. Thus there is a wide class of problems where rigorous results will be obtained with the use of computers.

This paper is the first in a series whose goal is the presentation of a criterion of stochasticity which can be checked by a computer. Here we consider the much simpler problem of the discovery by computers of closed orbits of systems of ordinary differential equations having the form

$$dx_i/dt = f_i(x_1, \dots, x_d), \qquad 1 \leqslant i \leqslant d \tag{1}$$

or, briefly, $dX/dt = F(X)$. This problem arises in the investigation of stochasticity in the Lorenz model because, according to the results of Afraimovich *et al.*,[4] the boundary of the stochastic attractor is defined on the base of stable manifolds of corresponding closed orbits.

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In Section 2 we formulate our main criterion for the existence of a closed orbit of the system (1) and present estimates of the coefficients which enter into the criterion. The computer is used to obtain these estimates and to verify the corresponding inequality. In Section 3 we prove the main criterion, and in Sections 4–7 we derive the estimations. The reader interested only in applications can omit these sections. In Section 8 we show the application of the criterion to the Lorenz model, proving rigorously the existence of a closed orbit for it for a special set of values of the parameters.

Now we give our main notations. X, Y, Z are d-dimensional vectors; their coordinates are denoted by lower case letters x_i, $1 \leqslant i \leqslant d$, or $x_i(X)$, $x_i(Y)$, $x_i(Z)$, and the norm $\|X\| = (\sum_{i=1}^{d} x_i^2)^{1/2}$. The one-parameter group of shifts along the trajectories of (1) is denoted by $\{S_t\}$. Let a be a fixed number; $\Gamma = \{X | x_d(X) = a\}$ is a hyperplane in R^d. Other notations are related to a neighborhood of a fixed interval γ of a trajectory (1) which begins at the point $X^{(0)} \in \Gamma$:

$$\gamma = \{X^{(0)}(t), \quad 0 \leqslant t \leqslant T\}, \qquad X^{(0)}(t) = S_t X^{(0)}(0) = S_t X^{(0)}$$

Namely, $W_\rho(\gamma)$ is the ρ-neighborhood of γ, and $U_\rho(X^{(0)})$ is the ρ-neighborhood of the point $X^{(0)}$ of the form

$$U_\rho(X^{(0)}) = \left\{X \middle| \sum_{i=1}^{d-1} [x_i(X) - x_i(X^{(0)})]^2 < \rho^2, \quad |x_d(X) - a| < \rho\right\}$$

$$U_\rho^{(1)}(X^{(0)}) = U_\rho(X^{(0)}) \cap \Gamma$$

$F'(X)$ is the matrix $\|\partial f_i(X)/\partial x_j\|$. The value of ρ depends on the problem under consideration. We shall consider the linearized system corresponding to γ, namely $dZ/dt = F'(S_t X^{(0)})Z$. We shall denote by $\mathscr{L}(t_1, t_2)$ the fundamental matrix of solutions of this system on the interval $t_1 \leqslant t \leqslant t_2$. We put

$$C_1 = \max_{0 \leqslant t_1 \leqslant t_2 \leqslant T} \|\mathscr{L}(t_1, t_2)\|$$

Suppose that the terms on the right-hand side of (1) are such that one can find a constant C_2 for which

$$\sum_{i,j,k} \left|\frac{\partial^2 f_i(X)}{\partial x_j \, \partial x_k} y_j z_k\right| \leqslant C_2 \|Y\| \, \|Z\|, \qquad X \in W_\rho(\gamma)$$

This constant always exists if the f_i are polynomials of powers not more than two. Further, let

$$C_3 = \inf_{X \in U_\rho(X^{(0)})} |f_d(X)|, \qquad C_4 = \max_{X \in U_\rho(X^{(0)})} \max_i |f_i(X)|$$

$$C_5 = \max_{X \in U_\rho(X^{(0)})} \|F'(X)\|, \qquad C_6 = \max_{X \in U_\rho(X^{(0)})} \|F(X)\|$$

2. MAIN CRITERION FOR THE EXISTENCE OF A CLOSED ORBIT OF (1) AND ITS VERIFICATION BY COMPUTER

We begin with a theoretical criterion for the existence of a closed orbit of (1). Assume that the terms on the right-hand side of (1) are C^∞ functions. Choose a hyperplane Γ and an initial point $X^{(0)} = \{x_1^{(0)},\ldots, x_d^{(0)}\} \in \Gamma$. We suppose that for some $T > 0$ the point $S_T X^{(0)} = X^{(1)} \in \Gamma$, $\epsilon = \|X^{(1)} - X^{(0)}\|$, and in a sufficiently small neighborhood $U \subset \Gamma$ of the point $X^{(0)}$ a Poincaré C^∞ mapping is defined which transforms a point $X \in U$ into a point $PX \in \Gamma$ with the help of the shift along the trajectory of the point X, $PX^{(0)} = X^{(1)}$. We can expand P in a Taylor series in the neighborhood of the point $X^{(0)}$. We put

$$Y = X - X^{(0)}, \qquad Q(Y) = P(X) - X^{(0)}, \qquad X \in U$$

and write Q in the form

$$Q(Y) = Y^{(0)} + LY + Q_1(Y)$$

Here $Y^{(0)} = X^{(1)} - X^{(0)}$, L is the matrix of the linear part of Q at the point $Y = 0$, and Q_1 is a nonlinear correction term. Suppose that the mapping Q satisfies the following condition: there exist positive constants ρ_o, K_0 such that for an arbitrary $\rho \leqslant \rho_0$ and arbitrary Y', Y'', $\|Y'\| \leqslant \rho$, $\|Y''\| \leqslant \rho$

$$\|Q_1(Y') - Q_1(Y'')\| \leqslant K_0 \rho \|Y' - Y''\|$$

This inequality expresses the quadratic character of Q_1.

Criterion. Let $\|Y^{(0)}\| = \epsilon$ and for some $\bar\rho_0 \leqslant \rho_0$

$$\|(L - E)^{-1}\|(\epsilon/\bar\rho_0 + K_0\bar\rho_0) \leqslant 1$$

Then in the $\bar\rho_0$ neighborhood of $X^{(0)}$ there exists one and only one fixed point of Q.

We shall use a computer to verify the validity of the last inequality. First we shall write down the estimations of all the numbers which enter into the main inequality. These estimations have been obtained under the assumption that the terms on the right-hand side in (1) are polynomials of powers not more than two. This is the most frequent case in applications.

Estimation of ϵ. Consider a finite-difference method with step $\Delta t = \Delta$: $X_{i+1} = RX_i$, where R is a transformation which is an approximation of S_Δ. In fact we get a sequence of points $X_0 = X^{(0)}$, X_1, $X_2,\ldots, X_n$, $\|X_{i+1} - RX_i\| \leqslant \alpha$. The value of α depends upon the precision with which the calculations are performed and is the only parameter that takes into account the properties of the computer that is used. In the general theory of dynamical systems a sequence of points X_i, $0 \leqslant i \leqslant n$, $\|X_{i+1} - RX_i\| \leqslant \alpha$, is called a pseudotrajectory (see the paper by Bowen[5] on the role of this concept in differential dynamics).

In the case under consideration the calculations have been performed until the corresponding intersection of the pseudotrajectory with the hyperplane Γ. Let $x_d(X_{n-1}) \geqslant a$ and $x_d(X_n) < a$ for some n. Using the usual linear interpolation we can find the point $\bar{X}$ for which $x_d(\bar{X}) = a$. If $P(X^{(0)}) = X^{(1)}$, we have

$$\epsilon = \|X^{(1)} - X^{(0)}\| \leqslant \|\bar{X} - X^{(1)}\| + \|\bar{X} - X^{(0)}\| \tag{2}$$

The number $\|\bar{X} - X^{(0)}\|$ is found from the results of calculations. We can write for the first term in (2)

$$\|\bar{X} - X^{(1)}\| \leqslant \|X^{(1)} - S_{n\Delta}X^{(0)} - (\bar{X} - X_n)\| + \|S_{n\Delta}X^{(0)} - X_n\|$$

The number $\|S_{n\Delta}X^{(0)} - X_n\|$ is an error which appears as the result of the difference method used. In Section 5 we derive an estimate of this error in the case of the difference method convenient for systems (1) where the terms on the right-hand are polynomials of second power. This estimate takes the form

$$\|S_{n\Delta}X^{(0)} - X_n\| \leqslant C_1 n\Delta^2 A \tag{3}$$

where A is the least root of the quadratic equation

$$A - 2(n\Delta)^2\Delta C_1{}^3 C_2 A^2 = \alpha\Delta^{-2} + \Delta[\tfrac{1}{5}(C_5\sqrt{d})^3 + \tfrac{1}{3}C_2 C_6{}^2] \tag{4}$$

The estimation of the difference $\|X^{(1)} - S_{n\Delta}X^{(0)} - (\bar{X} - X_n)\|$ is done explicitly.

Estimation of matrix elements of L. Let $l_{ik}(T)$ be matrix elements of the matrices $\mathscr{L}(0, T)$. Then

$$l_{ik} = l_{ik}(T) - \frac{f_i(S_T X^{(0)})}{f_d(S_T X^{(0)})}\, l_{dk}(T) \tag{5}$$

Therefore in order to find l_{ik} it is sufficient to determine $l_{ik}(T)$ by computer. This can be done most simply as follows. Let us take our pseudotrajectory $X_0, \ldots, X_n$. For every point X_i we construct the matrices $F'(X_i)$ and $\bar{\mathscr{L}}(0, i\Delta)$, where

$$\bar{\mathscr{L}}(0, (i+1)\Delta) = [E + \Delta F'(X_i)]\bar{\mathscr{L}}(0, i\Delta) + \delta\mathscr{L}_{i+1} \tag{6}$$

$\delta\mathscr{L}_{i+1}$ is an error which appears as the result of our approximation procedure, $\|\delta\mathscr{L}_{i+1}\| \leqslant \beta$, where β takes into account the precision of the calculation. The matrix $\bar{\mathscr{L}}(0, n\Delta)$ can be considered as an approximate value of $\mathscr{L}(0, T)$. In Section 7, [expression (19)] we derive an estimate of the difference $\bar{\mathscr{L}}(0, n\Delta) - \mathscr{L}(0, T)$:

$$\begin{aligned} &\|\bar{\mathscr{L}}(0, n\Delta) - \mathscr{L}(0, T)\| \\ &\quad \leqslant \{(1 + 2C_1)C_{10}\Delta + (C_1 C_3^{-1} C_5 + C_1 C_3^{-2} C_4 C_5) \\ &\qquad \times [2(C_1 TA + C_3^{-1})C_3^{-1} C_6 + C_1 TA]\}\Delta \end{aligned} \tag{7}$$

where $C_{10} = C_1(C_1{}^2C_2T^2A + C_1C_4{}^2T\Delta + \beta\Delta^{-1}T)$. It is worthwhile to note that there is another method for the definition of l_{ik} which is similar to the method of numerical differentiation. We do not write down the corresponding estimates.

The values of ϵ and the matrix L depend only upon the part of the trajectory of the point $X^{(0)}$. The constant K_0 depends upon properties of the dynamical system in the whole neighborhood $W_\rho(\gamma)$.

Estimation of K_0. For K_0 the following estimate is valid:

$$K_0 \leqslant d[B_1(C_1{}^2 + \tfrac{1}{2}) + C_1C_5(C_1 + B_1\rho) + C_7 + C_3^{-1}C_4(C_1C_5C_9\sqrt{d} + C_8C_9 + B_2) + C_1C_5C_9\sqrt{d} + C_8C_9] \quad (8)$$

Here we denote

$$B_1 = 2C_1{}^3C_2n\Delta, \qquad B_2 = 2C_1C_2(C_1{}^2 + \tfrac{1}{2})n\Delta$$

$$C_7 = C_1C_3^{-1}C_5(C_4 + C_6 + C_3^{-1}C_4C_6)(C_1 + B_1\rho)$$

$$C_8 = C_5B_1\rho + \tfrac{1}{2}C_1C_2\rho + C_1C_2B_1\rho^2 + \tfrac{1}{2}B_1C_2\rho^3$$

$$C_9 = [C_1 + 2B_1(2C_1{}^2 + 1)][C_3 - \rho_1(C_1C_5 + C_8\rho_1)]^{-1}$$

This estimate is derived in Section 6.

Additive inequalities. All the estimates were derived under the assumptions that

$$\exp(C_5\sqrt{d}\Delta) - [1 + C_5\sqrt{d}\Delta + \tfrac{1}{2}(C_5\sqrt{d}\Delta)^2] \leqslant \tfrac{1}{5}(C_5\sqrt{d}\Delta)^3$$

and

$$\exp(C_4\Delta) - 1 - C_4\Delta \leqslant C_4{}^2\Delta^2$$

where Δ is the time step.

Use of the estimate. First we find a point $X^{(0)}$ and a sequence of its pseudotrajectory the last point of which X_n leads to the point $\bar{X}$ which is very close to $X^{(0)}$. This is the only part where a high precision is needed in the calculations (in the example of Section 8 the norm $\|\bar{X} - X^{(0)}\|$ is of the order of 10^{-10}). Next we choose ρ and roughly estimate constants C_i, $2 \leqslant i \leqslant 6$. The estimation of C_1 again requires the use of a computer. To do this we take matrices $F'(X_i)$ and for all $t_1, t_2, t_1 = k\Delta_1, t_2 = l\Delta_1$ (k and l integers), $\Delta_1 > \Delta$, we find the matrices $\bar{\mathscr{L}}(t_1, t_2)$, where

$$\bar{\mathscr{L}}(t_1, (i+1)\Delta) = [E + \Delta\cdot F'(X_i)]\bar{\mathscr{L}}(t_1, i\Delta) + \delta\mathscr{L}_{i+1}$$

[see (6)] and estimate all norms $\|\bar{\mathscr{L}}(t_1, t_2)\|$. The constant C_1 can be estimated through the maximum of all these norms (see Section 8 for details).

Having C_i, $1 \leqslant i \leqslant 6$, and n we can estimate $T \leqslant n\Delta$ and the value $\rho_0 = \kappa\rho$, where

$$\kappa < \tfrac{1}{2}(C_3^{-1}C^4 + 1)^{-1}[C_1 + 2C_1C_2(n\Delta + 2\rho C_3^{-1})]^{-1}$$

Next we can determine the boundaries for each matrix element l_{ik}. This allows us to estimate the precision with which we find matrices $L - E$ and $(L - E)^{-1}$ and to get the estimate of the norm $\|(L - E)^{-1}\|$. Further we estimate K_0 with the help of (8) and determine the value $\bar{\rho}_0$ that enters into the criterion. If the main inequality of the criterion is valid, we can conclude that the closed orbit of the system (1) exists in the $\bar{\rho}_0$ neighborhood of $X^{(0)}$.

3. PROOF OF THE MAIN CRITERION

The main criterion was formulated in Section 2. Here we give a proof based upon a Newton method (see, for example, Ref. 6). This proof is due to N. N. Chentzova. It is simpler than our original proof.

We put $GY = QY - Y$ and $L_1 = L - E$. The fixed point of the mapping Q is a solution of the equation $GY = 0$. We look for Y by the method of successive approximations. Put $Y_0 = 0$, $Y_{k+1} = Y_k - L_1^{-1}(GY_k)$. We have

$$\begin{aligned} Y_{k+1} &= Y_k - L_1^{-1}(QY_k - Y_k) \\ &= Y_k - L_1^{-1}(Y^{(0)} + LY_k - Y_k + Q_1 Y_k) \\ &= -L_1^{-1} Y^{(0)} - L_1^{-1} Q_1 Y_k \end{aligned}$$

From this

$$\|Y_{k+1} - Y_k\| = \|L_1^{-1}[Q_1(Y_k) - Q_1(Y_{k-1})]\| \leqslant \|L_1^{-1}\| \, \|Q_1(Y_k) - Q_1(Y_{k-1})\|$$

Assume that all Y_i, $0 \leqslant i \leqslant k$, satisfy the inequality $\|Y_i\| \leqslant \bar{\rho}_0$. From the main inequality of the criterion we have

$$\|Q_1(Y_k) - Q_1(Y_{k-1})\| \leqslant K_0\bar{\rho}_0\|Y_k - Y_{k-1}\|$$

$$\|Y_{k+1} - Y_k\| \leqslant \|L_1^{-1}\|K_0\bar{\rho}_0\|Y_k - Y_{k-1}\| \leqslant (K_0\bar{\rho}_0\|L_1^{-1}\|)^k$$

$$\|Y_1 - Y_0\|, \qquad Y_1 = -L_1^{-1}Y^{(0)}, \qquad \|Y_1\| \leqslant \|L_1^{-1}\|\epsilon$$

and therefore

$$\|Y_{k+1}\| \leqslant \sum_{i=0}^{k} \|Y_{i+1} - Y_i\| \leqslant \|L_1^{-1}\|\epsilon \sum_{i=0}^{k} (\|L_1^{-1}\|K_0\bar{\rho}_0)^i$$

$$\leqslant \|L_1^{-1}\|\epsilon(1 - \|L_1^{-1}\|K_0\bar{\rho}_0)^{-1} \leqslant \bar{\rho}_0$$

Now we have $\|Y_k\| \leqslant \bar{\rho}_0$ for all k and the limit $\lim_{k\to\infty} Y_k = \bar{Y}$ exists. It is obvious that $G\bar{Y} = 0$ or $Q\bar{Y} = \bar{Y}$.

Unicity. Supposing that there exists another point $\bar{\bar{Y}}$, $\|\bar{\bar{Y}}\| \leqslant \bar{\rho}_0$, for which $G\bar{\bar{Y}} = 0$; we have

$$0 = G\bar{Y} - G\bar{\bar{Y}}, \qquad L_1(\bar{Y} - \bar{\bar{Y}}) = Q_1(\bar{Y}) - Q_1(\bar{\bar{Y}})$$

Therefore

$$\|\bar{Y} - \bar{\bar{Y}}\| = \|L_1^{-1}[Q_1(\bar{Y}) - Q_1(\bar{\bar{Y}})]\| \leqslant \|L_1^{-1}\|K_0\bar{\rho}_0\|\bar{Y} - \bar{\bar{Y}}\|$$

From the main inequality $\|L_1^{-1}\|K_0\bar{\rho}_0 < 1$. Therefore $\|\bar{Y} - \bar{\bar{Y}}\| = 0$. QED

4. AN ESTIMATE OF THE DIFFERENCE BETWEEN A SOLUTION OF (1) AND A SOLUTION OBTAINED THROUGH THE LINEARIZED EQUATION

The results of this section are valid for systems (1) where the terms on the right-hand side are polynomials of the second power. Let $\{S_t X^{(0)}, 0 \leqslant t \leqslant T\}$ be an interval of a trajectory of (1). The corresponding linearized system has the form

$$dZ/dt = F'(S_t X^{(0)})Z \tag{9}$$

We denote by $\mathscr{L}(t_1, t_2)$ the fundamental matrix of solutions of this system on the interval (t_1, t_2). As before

$$C_1 = \max_{0 \leqslant t_1 \leqslant t_2 \leqslant T} \|\mathscr{L}(t_1, t_2)\|$$

Let us take a point X which is close to $X^{(0)}$, $Y(0) = X - X^{(0)}$, $Y(t) = \mathscr{L}(0, t)Y(0)$. In this section we investigate the difference $S_t X - S_t X^{(0)} - Y(t) = \delta_1 X(t)$. Let us put also $X(t) = S_t X$, $X^{(0)}(t) = S_t X^{(0)}$, $\delta X(t) = X(t) - X^{(0)}(t)$.

Theorem 1. Let $\|Y(0)\| \leqslant \rho$, where ρ satisfies the inequality $\rho < (2TC_1{}^2C_2)^{-1/2}$. Then for $B_1(t) = 2C_1{}^3C_2 t$

$$\|S_t X - S_t X^{(0)} - Y(t)\| \leqslant B_1(t)\|Y(0)\|^2, \qquad 0 \leqslant t \leqslant T$$

Proof. We have

$$\delta_1 X(t) = \int_0^t \mathscr{L}(s, t)[\tfrac{1}{2}(F''Y(s), Y(s)) + (F''Y(s), \delta_1 X(s)) + \tfrac{1}{2}(F''\delta_1 X(s), \delta_1 X(s))]\, ds \tag{10}$$

We have used $F'' = \text{const}$ and the fact that $F'(X)$ is a linear function of X. Let $\mathscr{D}(t) = \max_{0 \leqslant t_1 \leqslant t} \|\delta_1 X(t_1)\|$. From the last equation we have for s, $0 \leqslant s \leqslant t$,

$$\|\delta_1 X(s)\| \leqslant C_1 s[\tfrac{1}{2}C_1{}^2C_2\|Y(0)\|^2 + C_1C_2\|Y(0)\|\mathscr{D}(t) + \tfrac{1}{2}C_2\mathscr{D}^2(t)] \tag{11}$$

Suppose that $\mathscr{D}_1 = \mathscr{D}_1(t)$ is the least root of the quadratic equation

$$tC_1[\tfrac{1}{2}C_1{}^2C_2\|Y(0)\|^2 + C_1C_2\|Y(0)\|\mathscr{D}_i + \tfrac{1}{2}C_2\mathscr{D}_1{}^2] = \mathscr{D}_1$$

We shall show that $\mathscr{D}(t) \leqslant \mathscr{D}_1(t)$. As a matter of fact, for sufficiently small s we have $\mathscr{D}(s) \leqslant \mathscr{D}_1(t)$. Let the equality $\mathscr{D}(s_0) = \mathscr{D}_1(t)$ for some $s_0 < t$ be valid. Then from (11)

$$s_0 C_1[\tfrac{1}{2}C_1{}^2C_2\|Y(0)\|^2 + C_1C_2\|Y(0)\|\mathscr{D}_1(t) + \tfrac{1}{2}C_2\mathscr{D}_1{}^2(t)] < \mathscr{D}_1$$

because $s_0 < t$, i.e., the equality $\mathscr{D}(s_0) = \mathscr{D}_1(t)$ is impossible. Therefore for all s, $0 \leqslant s \leqslant t$, we have $\mathscr{D}(s) \leqslant \mathscr{D}_1(t)$. Further,

$$\mathscr{D}_1(t) = \left(\frac{1}{tC_2} - C_1{}^2\|Y(0)\|^2\right) - \left[\left(\frac{1}{tC_2} - C_1{}^2\|Y(0)\|^2\right)^2 - C_1{}^3\|Y(0)\|^2\right]^{1/2}$$
$$\leqslant C_1{}^3\|Y(0)\|^2\left(\frac{1}{tC_2} - C_1{}^2\|Y(0)\|^2\right)^{-1} \leqslant 2tC_1{}^3C_2\|Y(0)\|^2$$

because of the assumption concerning ρ. QED

Theorem 2. Let

$$\|Y(0)\| \leqslant \rho = \min([2TC_1{}^2C_2]^{-1/2}, [4C_1C_2]^{-1}, \{2[2TC_1C_2]^{-1/2}\}^{-1})$$

Then, using the notations of Theorem 1 and putting $B_2(t) = 2tC_1C_2(\tfrac{1}{2} + C_1{}^2)$, we have

$$\|(\partial/\partial y_j)(\delta_1 X(t))\| \leqslant B_2(t)\|Y(0)\|$$

Proof. Differentiating both sides of (10) with respect to y_j and using $F'' = \text{const}$, we get

$$\frac{\partial}{\partial y_j}(\delta_i X) = \int_0^t \mathscr{L}(s,t)\left\{\left(F''\frac{\partial Y(s)}{\partial y_j}, Y(s)\right) + \left(F''\frac{\partial Y(s)}{\partial y_j}, \delta_1 X(s)\right)\right.$$
$$\left. + \left(F''Y(s), \frac{\partial}{\partial y_j}[\delta_1 X(s)]\right) + \left(F''\delta_1 X(s), \frac{\partial}{\partial y_j}[\delta_1 X(s)]\right)\right\} ds$$

Supposing

$$\mathscr{D}_j{}^*(t) = \max_{0 \leqslant s \leqslant t} \|(\partial/\partial y_j)(\delta_1 X(s))\|$$

we can write

$$\left\|\frac{\partial}{\partial y_j}(\delta_1 X)\right\| \leqslant C_1 t[C_1{}^2C_2\|Y(0)\| + 2tC_1{}^2C_2{}^2\|Y(0)\|^2$$
$$+ C_1C_2\|Y(0)\|\mathscr{D}_j{}^*(t) + 2tC_1C_2{}^2\mathscr{D}_j{}^*(t)\|Y(0)\|^2] \leqslant \mathscr{D}_j{}^*(t)$$

using the fact that $Y(s)$ is a linear function of y_j and $\|\partial y(s)/\partial y_j\| \leqslant C_1$. Further considerations as in Theorem 1 lead to

$$\mathscr{D}_j{}^*(t) \leqslant \frac{C_1{}^3C_2 t + 2tC_1{}^2C_2{}^2\|Y(0)\|}{1 - C_1C_2\|Y(0)\| - 2tC_1C_2{}^2\|Y(0)\|^2}\|Y(0)\|$$
$$\leqslant 2tC_1C_2(C_1{}^2 + \tfrac{1}{2})\|Y(0)\|$$

QED.

5. ESTIMATION OF THE ERROR IN THE FINITE-DIFFERENCE METHOD

In this section we describe the finite-difference method which is convenient for systems of type (1), where the terms on the right-hand side are polynomials of power not higher than two, and investigate the error which arises when this method is applied.

We write $f_i(x_1,..., x_d) = f_i(X)$ in the form $f_i(X) = (l_i, X) + (B_iX, X)$, where (l_i, X) is linear form and (B_iX, X) is quadratic form; in vector notation $F(X) = (l, X) + (BX, X)$. We replace system (1) by the system of integral equations

$$X(t) = S_tX(0) = X(0) + \int_0^t F(X(s))\, ds$$

i.e., in coordinates,

$$x_i(t) = x_i(0) + \int_0^t f_i(X(s))\, ds, \qquad 1 \leqslant i \leqslant d$$

In using the method of successive approximations to solve this equation take as the zeroth approximation $X_0(t) \equiv X(0)$; then the first approximation

$$X_1(t) = X(0) + \int_0^t F(X(0))\, ds = X(0) + tF(X(0))$$

leads to the usual Euler method. Consider the second approximation:

$$\begin{aligned} X_2(t) &= X(0) + \int_0^t F(X_1(s))\, ds \\ &= X(0) + \int_0^t [(l, X(0)) + s(l, X(0)) + s(BX(0), X(0)) \\ &\quad + (BX(0), X(0)) \\ &\quad + 2s(BX(0), F(X(0))) + s^2(BF(X(0)), F(X(0)))]\, ds \\ &= X(0) + t[(l, X(0)) + (BX(0), X(0))] \\ &\quad + t^2[\tfrac{1}{2}(l, F(X(0))) + (BX(0), F(X(0)))] + \tfrac{1}{3}t^3(BF(X(0)), F(X(0))) \end{aligned}$$

The method of finite differences with the step Δ which we have used transforms the point X into the point RX, where

$$RX = X + \Delta[(l, X) + (BX, X)] + \tfrac{1}{2}\Delta^2[(l, F(X)) + (BX, F(X))]$$

Let $X_0, X_1,..., X_n$ be a pseudotrajectory of the length $n + 1$, $T = n\Delta$, i.e., $\|X_{i+1} - RX_i\| \leqslant \alpha$. We shall estimate the norm $\|S_TX_0 - X_n\|$. The following considerations were used by Losinsky.[7] Let us put $Z_k = X_k - S_{k\Delta}X_0$. The vector Z_k characterizes the error at the moment $k\Delta$.

We consider the system of equations in variations along the trajectory of the point

$$dZ/dt = F'(S_t X_0)Z \tag{12}$$

We denote by $\mathscr{L}(t_1, t_2)$ the fundamental matrix of solutions of the system (12) on the interval (t_1, t_2). We make the inductive assumption $Z_k = \sum_{j=0}^{k} \mathscr{L}(j\Delta, k\Delta)V_j$ and look for recurrence equations for V_j. We have

$$Z_{k+1} = X_{k+1} - S_{(k+1)\Delta}X_0 = X_{k+1} - S_\Delta X_k + (S_\Delta X_k - S_{(k+1)\Delta}X_0)$$

For the second difference we have

$$\begin{aligned} S_\Delta X_k - S_{(k+1)\Delta}X_0 &= S_\Delta(Z_k + S_{k\Delta}X_0) - S_\Delta(S_{k\Delta}X_0) \\ &= \mathscr{L}(k\Delta, (k+1)\Delta)Z_k + \delta_1 Z_k \\ &= \sum_{j=0}^{k} \mathscr{L}(j\Delta, (k+1)\Delta)V_j + \delta_1 Z_k \end{aligned}$$

Put $V_{k+1} = X_{k+1} - S_\Delta X_k + \delta_1 Z_k$.

From the standard estimations of the method of successive approximations we have

$$\begin{aligned} \|X_{k+1} - S_\Delta X_k\| &= \|X_{k+1} - RX_k\| + \|RX_k - S_\Delta X_k\| \\ &\leqslant \alpha + \exp(C_5\sqrt{d}\Delta) - [1 + C_5\sqrt{d}\Delta + \tfrac{1}{2}(C_5\sqrt{d}\Delta)^2] \\ &\quad + \tfrac{1}{3}C_2{C_6}^2\Delta^3 \end{aligned}$$

where $C_2{C_6}^2 \geqslant \max_{X\in W} \|(BF(X), F(X))\|$. Let us suppose that Δ is so small that

$$\exp(C_5\sqrt{d}\Delta) - [1 + C_5\sqrt{d}\Delta + \tfrac{1}{2}(C_5\sqrt{d}\Delta)^2] \leqslant \tfrac{1}{5}(C_5\sqrt{d})^3\Delta^3$$

Then

$$\|X_{k+1} - S_\Delta X_k\| \leqslant \alpha + [\tfrac{1}{5}(C_5\sqrt{d})^3 + \tfrac{1}{3}C_2{C_6}^2]\Delta^3$$

From Theorem 1 we have

$$\|\delta_1 Z_k\| \leqslant B_1(\Delta){C_1}^2\left[\sum_{j=0}^{k} \|V_j\|\right]^2$$

Now we make the inductive hypothesis $\|V_j\| \leqslant A\Delta^2$, $0 \leqslant j \leqslant k$, and we shall find a condition on A under which the inequality is also valid for $j = k + 1$. We have

$$\|V_{k+1}\| \leqslant \|\delta_1 Z_k\| + \|X_{k+1} - S_\Delta X_k\|$$

$$\leqslant B_1(\Delta)C_1{}^2(k+1)^2A^2\Delta^4 + \Delta^2\left[\frac{\alpha}{\Delta^2} + \Delta\left(\frac{(C_5\sqrt{d})^3}{5} + \frac{C_2C_6{}^2}{3}\right)\right]$$

$$= \Delta^2\left\{\frac{\alpha}{\Delta^2} + \Delta\left[\frac{(C_5\sqrt{d})^3}{5} + \frac{C_2C_6{}^2}{3}\right] + B_1(\Delta)C_1{}^2T^2A^2\right\}$$

It can be seen from Theorem 1 that $B_1(\Delta)$ is proportional to Δ. Therefore if

$$A - T^2C_1{}^2B_1(\Delta)A^2 \geqslant \frac{\alpha}{\Delta^2} + \Delta\left[\frac{(C_5\sqrt{d})^3}{5} + \frac{C_2C_6{}^2}{3}\right] \tag{13}$$

we get $\|V_{k+1}\| \leqslant A\Delta^2$, i.e.,

$$\|Z_k\| \leqslant C_1TA\Delta \tag{14}$$

This is the final estimate and coincides with (3). If we take the equality in (3) we get the explicit expression (4) for A.

6. ESTIMATION OF THE CONSTANT K_0

In this section we shall derive an estimate for K_0 which takes into account the properties of the nonlinear correction term Q_1. As in Section 2, we consider the plane Γ and the Poincaré mapping P of the neighborhood $\bar{U}_\rho(X^{(0)}) \subset \Gamma$. Let $PX^{(0)} = X^{(1)} = S_TX^{(0)}$. Let us denote by l_{ik} the matrix elements of the matrix L. For every $X \in \bar{U}_\rho(X^{(0)})$ the time $\hat{t} = \hat{t}(X)$ for the point $X = X^{(0)} + Y$ to move to the plane Γ can be found from the equation

$$x_d(X^{(0)}(\hat{t}) + Y(\hat{t}) + \delta_1X(\hat{t})) = x_d(X^{(0)}(\hat{t})) + x_d(Y(\hat{t})) + x_d(\delta_1X(t)) = a \tag{15}$$

Let $\kappa > 0$ be such that $\kappa(C_1 + B_1\rho) < \frac{1}{2}(1 + C_3^{-1}C_4)^{-1}$, $B_1 = B_1(T_1)$, $T_1 = T + 2\rho C_3^{-1}$.

Theorem 3. For every point $X \in \bar{U}_{\kappa\rho}(X^{(0)})$:

1. The interval of the trajectory $\{S_tX, 0 \leqslant t \leqslant T_1\}$ is contained in the $\frac{1}{2}\rho$-neighborhood of the trajectory $\{S_tX^{(0)}, 0 \leqslant t \leqslant T_1\}$.

2. The Poincaré mapping P is continuous in $\bar{\bar{U}}_{\kappa\rho}(X^{(0)})$ and

$$|\hat{t}(X) - T| \leqslant C_3^{-1}(C_1 + B_1\|Y\|)\|Y\|$$

3. For $\epsilon = \|PX^{(0)} - X^{(0)}\| = \|S_TX^{(0)} - X^{(0)}\| \leqslant \frac{1}{2}\rho$,

$$PX \in \bar{U}_\rho(X^{(0)})$$

Proof. We have $S_tX = S_tX^{(0)} + Y(t) + \delta_1 X(t)$, and on the basis of Theorem 1 we get

$$\|S_tX - S_tX^{(0)}\| \leqslant \|Y(t)\| + \|\delta_1 X(t)\|$$
$$\leqslant (C_1 + B_1\|Y\|)\|Y\| \leqslant (C_1 + B_1\rho)\|Y\| \leqslant \kappa(C_1 + B_1\rho)\rho \leqslant \tfrac{1}{2}\rho$$

Thus the first statement is proved. Let us consider moments $T^-(X^{(0)}) = T^-$ and $T^+(X_0) = T^+$ such that $x_d(S_{T^-}X^{(0)}) = a + \frac{1}{4}C_3C_4^{-1}\rho$ and $x_d(S_{T^+}X^{(0)}) = a - C_3C_4^{-1}/4$. Then

$$|x_d(S_{T^-}X^{(0)}) - x_d(S_{T^-}X)| \leqslant (C_1 + B_1\rho)\|Y\|$$
$$|x_d(S_{T^+}X^{(0)}) - x_d(S_{T^+}X)| \leqslant (C_1 + B_1\rho)\|Y\|$$

Therefore

$$a + \tfrac{1}{4}C_3C_4^{-1}\rho - \kappa(C_1 + B_1\rho)\rho \leqslant x_d(S_{T^-}X) \leqslant a + \tfrac{1}{4}C_3C_4^{-1}\rho + \kappa(C_1 + B_1\rho)\rho$$
$$a - \tfrac{1}{4}C_3C_4^{-1}\rho - \kappa(C_1 + B_1\rho)\rho \leqslant x_d(S_{T^+}X) \leqslant a - \tfrac{1}{4}C_3C_4^{-1}\rho + \kappa(C_1 + B_1\rho)\rho$$

From the conditions concerning κ we have $T^- \leqslant \hat{t} = \hat{t}(X) \leqslant T^+$. Now we shall prove the second statement of the theorem. We have

$$C_3|\hat{t}(X) - T| = C_3|\hat{t}(X) - \hat{t}(X^{(0)})|$$
$$\leqslant |x_d(S_{\hat{t}}X^{(0)}) - x_d(S_tX^{(0)})| = |x_d(S_{\hat{t}}X^{(0)}) - a|$$
$$= |x_d(S_{\hat{t}}X^{(0)}) - x_d(S_{\hat{t}}X)| \leqslant (C_1 + B_1\|Y\|)\|Y\|$$

Thus

$$|\hat{t}(X) - T| \leqslant C_3^{-1}(C_1 + B_1\|Y\|)\|Y\|$$

Finally, for the third statement of the theorem we have

$$\|S_{\hat{t}}X - X^{(0)}\| \leqslant \|S_{\hat{t}}X - S_TX\| + \|S_TX - S_TX^{(0)}\| + \|S_TX^{(0)} - X^{(0)}\|$$
$$\leqslant C_4C_3^{-1}(C_1 + B_1\rho)\kappa\rho + \kappa(C_1 + B_1\rho)\rho + \epsilon$$
$$= \epsilon + \rho\kappa(C_1 + B_1\rho)(C_4C_3^{-1} + 1) < \rho$$

QED.

Now we consider the main problem, namely the estimation of the constant K_0. We write

$$Y^{(1)} = Q_1(X) = PX - L(X - X^{(0)}) - PX^{(0)} - X^{(0)}$$

Using (5) for $1 \leqslant j \leqslant d-1$, we have

$$y_j^{(1)} = x_j(Y^{(1)}) = x_j(S_{\hat{t}}X^{(0)}) - x_j(S_T X^{(0)}) + \frac{f_j(S_T X^{(0)})}{f_d(S_T X^{(0)})} \sum_{k=1}^{d-1} l_{dk}(T) y_k$$
$$+ \sum_{k=1}^{d-1} [l_{jk}(\hat{t}) - l_{jk}(T)] y_k + x_j(\delta_1 X(\hat{t})) + x_j(S_T X^{(0)} - X^{(0)})$$
$$y_k = x_k(Y), \qquad Y = X - X^{(0)}$$

Let us estimate the derivatives $\partial y_j^{(1)}/\partial y_l$. We can write

$$\frac{\partial y_j^{(1)}}{\partial y_l} = f_j(S_{\hat{t}}X^{(0)}) \frac{\partial \hat{t}}{\partial y_l} + \frac{f_j(S_T X^{(0)})}{f_d(S_T X^{(0)})} l_{dl}(T) + \sum_{k=1}^{d-1} \frac{d}{dt} l_{jk}(\hat{t}) \frac{\partial \hat{t}}{\partial y_l} y_k$$
$$+ \sum_{k=1}^{d-1} [l_{jl}(\hat{t}) - l_{jk}(T)] + \frac{\partial}{\partial t}[x_j(\delta_1 X(t))]|_{t=\hat{t}} \frac{\partial \hat{t}}{\partial y_l} + \frac{\partial x_j(\delta_1 X(\hat{t}))}{\partial y_l} \tag{16}$$

Differentiating both sides of (15) with respect to y_l, we get

$$f_d(S_{\hat{t}}X^{(0)}) \frac{\partial \hat{t}}{\partial y_l} + l_{dl}(\hat{t}) + \sum_{k=1}^{d-1} \frac{d}{dt}[l_{dk}(\hat{t})] y_k$$
$$+ \frac{d}{dt}[x_d(\delta_1 X(\hat{t}))] \frac{\partial \hat{t}}{\partial y_l} + \frac{\partial x_d(\delta_1 X(\hat{t}))}{\partial y_l} = 0 \tag{17}$$

Using (17), we get from (16) the expression which we shall estimate

$$\frac{\partial y_j^{(1)}}{\partial y_l} = \left[\frac{f_j(S_T X^{(0)})}{f_d(S_T X^{(0)})} l_{dl}(T) - \frac{f_j(S_{\hat{t}} X^{(0)})}{f_d(S_{\hat{t}} X^{(0)})} l_{dl}(\hat{t})\right]$$
$$- \frac{f_j(S_{\hat{t}} X^{(0)})}{f_d(S_{\hat{t}} X^{(0)})} \left[\left(\sum_{k=1}^{d-1} \frac{dl_{dk}}{dt}\bigg|_{t=\hat{t}} y_k + \frac{\partial x_d(\delta_1 X(\hat{t}))}{\partial t}\right) \frac{\partial \hat{t}}{\partial y_l} + \frac{\partial}{\partial y_l} x_d(\delta_1 X(\hat{t}))\right]$$
$$+ \frac{\partial \hat{t}}{\partial y_l} \sum_{k=1}^{d-1} \frac{d}{dt} l_{jk}|_{t=\hat{t}} y_k + [l_{jl}(\hat{t}) - l_{jl}(T)]$$
$$+ \left[\frac{\partial}{\partial t} x_j(\delta_1 X(t))|_{t=\hat{t}} \frac{\partial \hat{t}}{\partial y_l}\right] + \frac{\partial}{\partial y_l} x_j(\delta_1 X(\hat{t}))$$
$$= J_1 + J_2 + J_3 + J_4 + J_5 + J_6$$

From Theorem 2 we have

$$|J_6| \leqslant B_1(T_1)(C_1{}^2 + \tfrac{1}{2})\|Y\| \tag{18a}$$

From Theorem 3, part 2, we get

$$|J_4| \leqslant C_1 C_5 |T - \hat{t}| \leqslant C_1 C_5 (C_1 + B_1 \rho) \| Y \| \tag{18b}$$

$$\begin{aligned}
|J_1| &\leqslant \left| \frac{f_j(S_T X^{(0)})}{f_d(S_T X^{(0)})} l_{dl}(T) - \frac{f_j(S_{\hat{t}} X^{(0)})}{f_d(S_{\hat{t}} X^{(0)})} l_{dl}(\hat{t}) \right| \\
&\leqslant \frac{|f_j(S_T X^{(0)}) - f_j(S_{\hat{t}} X^{(0)})|}{|f_d(S_T X^{(0)})|} |l_{dl}(T)| + \frac{|f_j(S_{\hat{t}} X^{(0)}| \, |l_{dl}(T) - l_{dl}(\hat{t})|}{|f_d(S_T X^{(0)})|} \\
&\quad + \frac{|f_j(S_{\hat{t}} X^{(0)})| \, |l_{dl}(\hat{t})|}{|f_d(S_{\hat{t}} X^{(0)})| \, |f_d(S_T X^{(0)})|} |f_d(S_{\hat{t}} X^{(0)}) - f_d(S_T X^{(0)})| \\
&\leqslant (C_1 C_3^{-1} C_5 C_6 + C_1 C_3^{-1} C_4 C_5 + C_1 C_3^{-2} C_4 C_5 C_6) |T - \hat{t}| \\
&\leqslant C_1 C_3^{-1} C_5 (C_4 + C_6 + C_3^{-1} C_4 C_6)(C_1 + B_1 \rho) \| Y \| = C_7 \| Y \|
\end{aligned} \tag{18c}$$

Now we shall estimate the terms which contain $\partial \hat{t} / \partial y_l$. From (17) we have

$$\left| \frac{\partial \hat{t}}{\partial y_l} \right| \leqslant \frac{|l_{dl}(\hat{t})| + |(\partial / \partial y_l) x_d(\delta_1 X(\hat{t}))|}{|f_d(S_{\hat{t}} X^{(0)})| - \| Y \| [\sum_{k=1}^{d-1} |dl_{dk}(\hat{t})/dt| + |(d/dt) x_d(\delta_1 X(\hat{t}))|}$$

As before, $|l_{dl}(\hat{t})| \leqslant C_1$. Because of Theorem 2

$$\left| \frac{\partial}{\partial y_l} [x_d(\delta_1 X(t))] \right| \leqslant B_1(T_1)(C_1^2 + \tfrac{1}{2}) \| Y \|$$

$$\left| \frac{d}{dt} l_{dk}(\hat{t}) \right| \leqslant C_1 C_5$$

From (10) and Theorem 1

$$\left| \frac{\partial}{\partial t} x_d(\delta_1 X(t)) \right| \leqslant [C_5 B_1 \rho + \tfrac{1}{2} C_1 C_2 \rho + C_1 C_2 B_1 \rho^2 + \tfrac{1}{2} B_1^2 C_2 \rho^3] \| Y \| = C_8 \| Y \|$$

Thus

$$\left| \frac{\partial \hat{t}}{\partial y_l} \right| \leqslant \frac{C_1 + B_1(T_1)(2C_1^2 + 1)}{C_3 - \rho(C_1 C_5 + C_8 \rho)} = C_9$$

and

$$|J_2| \leqslant C_3^{-1} C_4 [C_1 C_5 C_9 \sqrt{d} + C_8 C_9 + B_2(T_1)] \| Y \| \tag{18d}$$

$$|J_3| \leqslant C_1 C_5 C_9 \sqrt{d} \, \| Y \| \tag{18e}$$

$$|J_5| \leqslant C_8 C_9 \| Y \| \tag{18f}$$

Collecting together the estimates (18a)–(18f), we get

$$|\partial y_j^{(1)} / \partial y_l| \leqslant K_1 \| Y \|$$

where

$$K_1 = B_1(T_1)(C_1{}^2 + \tfrac{1}{2}) + C_1C_5[C_1 + B_1(T_1)\rho] + C_7 + C_3^{-1}C_4[C_1C_5C_9\sqrt{d} + C_8C_9 + B_2(T_1)] + C_1C_5C_9\sqrt{d} + C_8C_9$$

Now, we can put $K_0 = dK_1$.

7. ESTIMATION OF THE MATRIX ELEMENTS OF THE LINEARIZATION OF THE POINCARÉ MAPPING WITH THE USE OF A COMPUTER

As in Section 2, let L denote the matrix of the linearization of the Poincaré mapping P at the point $X^{(0)}$. In this section we shall consider the precision with which the matrix elements of L can be found by computer. Using a computer we find approximately the matrix $\mathscr{L}(0, T)$ and, with (5), the matrix elements l_{ij} of the matrix L. As was mentioned in Section 2, the simplest way to find the matrix $\mathscr{L}(0, T)$ consists in the following. We consider a pseudotrajectory $X_0, X_1, \ldots, X_n$ as before. For every point X_i we construct matrices $F'(X_i)$ and $\mathscr{L}(0, i\Delta)$ where

$$\bar{\mathscr{L}}(0, i\Delta) = [E + \Delta F'(X_{i-1})]\bar{\mathscr{L}}(0, (i-1)\Delta) + \delta\mathscr{L}_i$$

Here $\delta\mathscr{L}_i$ is the error arising from roundoff errors, $\|\delta\mathscr{L}_{i+1}\| \leqslant \beta$. Then $\bar{\mathscr{L}}(0, n\Delta)$ is the approximate value of the matrix $\mathscr{L}(0, n\Delta)$. In order to estimate the error, we write

$$\begin{aligned}
&\mathscr{L}(0, (i+1)\Delta) - \bar{\mathscr{L}}(0, (i+1)\Delta)\\
&\quad = [E + \Delta F'(X_i)][\mathscr{L}(0, i\Delta) - \bar{\mathscr{L}}(0, i\Delta)]\\
&\qquad + \Delta[F'(X_i) - F'(S_{i\Delta}X_0)]\mathscr{L}(0, i\Delta)\\
&\qquad + [E + \Delta F'(S_{i\Delta}X_0)]\mathscr{L}(0, i\Delta) - \mathscr{L}(0, (i+1)\Delta) + \delta\mathscr{L}_{i+1}\\
&\quad = [E + \Delta F'(X_i)][\mathscr{L}(0, i\Delta) - \bar{\mathscr{L}}(0, i\Delta)] + \delta_1\mathscr{L}_{i+1}
\end{aligned}$$

where

$$\begin{aligned}
\delta_1\mathscr{L}_{i+1} &= \Delta[F'(X_i) - F'(S_{i\Delta}X_0)]\mathscr{L}(0, i\Delta)\\
&\quad + [E + \Delta F'(S_{i\Delta}X_0)]\mathscr{L}(0, i\Delta) - \mathscr{L}(0, (i+1)\Delta) + \delta\mathscr{L}_{i+1}
\end{aligned}$$

Now, we can write

$$\mathscr{L}(0, (i+1)\Delta) - \bar{\mathscr{L}}(0, (i+1)\Delta) = \sum_{k=0}^{i} \prod_{j=k}^{i} [E + \Delta F'(X_j)]\, \delta_1\mathscr{L}_k$$

The constant C_1 also can be estimated approximately with the help of the norms of the products $\prod_{j=k}^{i} [E + \Delta F'(X_j)]$ (see Section 8). Thus we can use the inequality $\|\prod_{j=k}^{i} (E + \Delta F'(X_j))\| \leqslant C_1$ for arbitrary k, i. Then

$$\|\mathscr{L}(0, T) - \bar{\mathscr{L}}(0, n\Delta)\| \leqslant C_1 \sum_{k=0}^{i} \|\delta_1 \mathscr{L}_k\|$$

Let us estimate the norms $\|\delta_1 \mathscr{L}_k\|$. Using the linearity of F' and Theorem 1 [see (14)], we have

$$\begin{aligned} \|F'(X_i) - F'(S_{i\Delta} X_0)\| &= \|F''(X_i - S_{i\Delta} X_0)\| \\ &\leqslant C_2 \|X_i - S_{i\Delta} X_0\| \leqslant C_1 C_2 i \Delta^2 A \end{aligned}$$

In an analogous way

$$\begin{aligned} &\|[E + \Delta F'(S_{i\Delta} X_0)] \mathscr{L}(0, i\Delta) - \mathscr{L}(0, (i+1)\Delta)\| \\ &\quad = \|[E + \Delta F'(S_{i\Delta} X_0)] \mathscr{L}(0, i\Delta) - \mathscr{L}(i\Delta, (i+1)\Delta) \mathscr{L}(0, i\Delta)\| \\ &\quad \leqslant C_1 \|[E + \Delta F'(S_{i\Delta} X_0)] - \mathscr{L}(i\Delta, (i+1)\Delta)\| \leqslant C_1 C_4{}^2 \Delta^2 \end{aligned}$$

The last estimate is obtained by the standard method of successive approximations. It is valid for Δ sufficiently small that $\exp(C_4\Delta) - 1 - C_4\Delta \leqslant C_4{}^2\Delta^2$. Collecting together all the estimates, we get

$$\delta = \|\mathscr{L}(0, n\Delta) - \bar{\mathscr{L}}(0, n\Delta)\| \leqslant C_1[C_1{}^2 C_2 T^2 A + C_1 C_4{}^2 T\Delta + (\beta/\Delta)T] = C_{10}$$

This is the final estimate. Using this estimate and putting

$$\bar{l}_{ik} = \bar{l}_{ik}(n\Delta) - [\bar{l}_{ik}(n\Delta) - f_i(X_n)/f_d(X_n)]\bar{l}_{dk}(n\Delta)$$

where $\bar{l}_{ik}(n\Delta)$ are the matrix elements of the matrix $\bar{\mathscr{L}}(0, n\Delta)$, we get

$$\begin{aligned} |l_{ik} - \bar{l}_{ik}| &\leqslant \delta(1 + 2C_1) + (C_1 C_3^{-1} C_5 + C_1 C_3^{-2} C_4 C_5)|X_n - S_T X_0| \\ &\leqslant (1 + 2C_1)C_{10} + (C_1 C_3^{-1} C_5 + C_1 C_3^{-2} C_4 C_5) \\ &\quad \times [2C_3^{-1} C_6 (C_1 T A + C_3^{-1}) + C_1 T A]\Delta \end{aligned} \tag{19}$$

As mentioned before, the matrix elements l_{ik} also can be found by a procedure similar to the method of numerical differentiation.

8. APPLICATION TO THE LORENZ MODEL

Afraimovich *et al.*,[4] Guckenheimer,[8] and Williams[9] have presented theoretical and numerical investigations of the Lorenz model[1] which have shed light on effects discovered numerically by Lorenz. We believe that a method using a computer will have to be used in order to prove rigorous results for this model. We shall establish rigorously the existence of a closed orbit which according to Ref. 4 determines partly the boundary of the Lorenz attractor.

We have considered the system of three ordinary differential equations

$$\begin{aligned} dx/dt &= a_1x + b_1yz + b_1xz \\ dy/dt &= a_2y - b_1yz - b_1xz \\ dz/dt &= -a_3z + (x+y)(b_2x + b_3y) \end{aligned} \tag{20}$$

This system is obtained from the usual Lorenz system with the help of a linear change of variables. We used the following values of the parameters: $r = 28$, $\sigma = 6$, $b = 8/3$ (in the original notation of Lorenz).

According to Ref. 4, for these values of the parameters the stochastic attractor already exists. The corresponding values of the coefficients of the system (20) are

$$\begin{aligned} a_1 &= 9.700378782, & b_1 &= 0.227266206 \\ a_2 &= 16.700378782, & b_2 &= 2.616729797 \\ a_3 &= 8/3 = 2.666666667, & b_3 &= 1.783396463 \end{aligned}$$

We have considered the Poincaré mapping of the hyperplane $z = 27$. With the help of the method of trial and error the point $X^{(0)}$

$$x = 3.50078718468, \quad y = 3.33033178466, \quad z = 27$$

was found. Calculations made with a time step $\Delta = 10^{-5}$ in the difference method described in Section 5 led to the points

$$X_n = (3.5007926423;\ 3.3303411800;\ 27.0000342849)$$

$$X_{n+1} = (3.5007846842;\ 3.330327479;\ 26.9999842901)$$

The calculation was made with double precision. This makes it possible for us to take $\alpha = 10^{-15}$. By linear interpolation we obtain the point

$$\bar{X} = (3.500787119;\ 3.330331785;\ 27)$$

for which $\|X^{(0)} - \bar{X}\| \leqslant 2 \times 10^{-9}$.[2] As we shall see, such a high precision is needed for proving rigorous results.

Let us determine the constants C_i. We put $\rho = 0.001$. It is easy to check that we can take $C_2 = 6$, $C_3 = 50$, $C_4 = 100$, $C_5 = 17$, $C_6 = 110$.

The main problem for which the computer is needed again concerns the matrix L and the estimation of the constant C_1. To determine the constant C_1

[2] The same point and some constants related to it also were found by J. Ford and his collaborators. We use this occasion to express to them our sincere gratitude for their participation in this work.

we considered the sequence X_i, $0 \leqslant i \leqslant n$, which was obtained in the process of calculation. For each interval of the sequence $X_i, X_{i+1}, \ldots, X_{i+10^3-1}$, with $i = 10^3 j$ and $\Delta_1 = 10^{-2}$, where j is an integer, the matrix

$$\tilde{\mathscr{L}}(j\Delta_1, (j+1)\Delta_1) = \prod_{k=1}^{i+10^3-1} [E + 10^{-5} F'(X_k)]$$

was constructed and for every $j_1, j_2, j_1 < j_2$, we found the matrices

$$\tilde{\mathscr{L}}(j_1\Delta_1, j_2\Delta_1) = \prod_{j_1 \leqslant j < j_2} \tilde{\mathscr{L}}(j\Delta_1, (j+1)\Delta)$$

Next we estimated all norms $\|\tilde{\mathscr{L}}(j_1\Delta_1, j_2\Delta_1)\|$. We obtained

$$\tilde{C}_1 = \max_{j_1, j_2} \|\tilde{\mathscr{L}}(j_1\Delta_1, j_2\Delta_1)\| = 23$$

Estimation of the difference $|C_1 - \tilde{C}_1|$ is based upon inductive considerations. Let us denote

$$d_j = \max_{j_1, j_2 \leqslant j} \|\mathscr{L}(j_1\Delta_1, j_2\Delta_1) - \tilde{\mathscr{L}}(j_1\Delta_1, j_2\Delta_1)\|$$

Then for any $j_1 < j + 1$,

$$\begin{aligned}
&\|\mathscr{L}(j_1\Delta_1, (j+1)\Delta_1) - \tilde{\mathscr{L}}(j_1\Delta_1, (j+1)\Delta_1)\| \\
&\quad \leqslant \|\tilde{\mathscr{L}}(j_1\Delta_1, j\Delta_1)\| \, \|\tilde{\mathscr{L}}(j\Delta_1, (j+1)\Delta_1) - \mathscr{L}(j\Delta_1, (j+1)\Delta_1)\| \\
&\quad\quad + \|\mathscr{L}(j_1\Delta_1, j\Delta_1) - \tilde{\mathscr{L}}(j_1\Delta_1, j\Delta_1)\| \\
&\quad\quad \times \|\tilde{\mathscr{L}}(j\Delta_1, (j+1)\Delta_1) - \mathscr{L}(j\Delta_1, (j+1)\Delta_1)\| \\
&\quad\quad + \|\mathscr{L}(j_1\Delta_1, j\Delta_1) - \tilde{\mathscr{L}}(j_1\Delta_1, j\Delta)\| \, \|\tilde{\mathscr{L}}(j\Delta_1, (j+1)\Delta_1)\| \\
&\quad \leqslant (\tilde{C}_1 + d_j)\|\tilde{\mathscr{L}}(j\Delta_1, (j+1)\Delta_1) - \mathscr{L}(j\Delta_1, (j+1)\Delta_1)\| \\
&\quad\quad + d_j\|\tilde{\mathscr{L}}(j\Delta_1, (j+1)\Delta_1)\|
\end{aligned}$$

The value $\|\tilde{\mathscr{L}}(j\Delta_1, (j+1)\Delta_1\|$ also can be found from numerical calculations on a computer. In our case it turned out that $\|\tilde{\mathscr{L}}(j\Delta_1, (j+1)\Delta_1\| \leqslant 3$.

Now we must consider

$$\|\tilde{\mathscr{L}}(j\Delta_1, (j+1)\Delta_1) - \mathscr{L}(j\Delta_1, (j+1)\Delta_1)\|$$

We have

$$\|\tilde{\mathscr{L}}(j\Delta_1, (j+1)\Delta_1) - \mathscr{L}(j\Delta_1, (j+1)\Delta_1)\|$$

$$\leqslant \|\mathscr{L}(j\Delta_1, (j+1)\Delta_1) - \prod_{k=j10^3}^{(j+1)10^3-1} [E + 10^{-5}F'(S_{k10^{-5}}X^{(0)})]\|$$

$$+ \Delta \sum_{k=j10^3}^{(j+1)10^3-1} \left\| \prod_{l_1<k} [E + 10^{-5}F'(S_{l_1 10^{-5}}X^{(0)})] \right.$$

$$\times [F'(S_{k10^{-5}}X^{(0)}) - F'(X_k)]$$

$$\left. \times \prod_{l_2>k} [E + 10^{-5}F'(S_{l_2 10^{-5}}X^{(0)})] \right\|$$

Let us estimate each term separately. We have

$$\mathscr{L}(j\Delta_1, (j+1)\Delta_1) - \prod_{k=j10^3}^{(j+1)10^3-1} [E + 10^{-5}F'(S_{k10^{-5}}X^{(0)})]$$

$$= \sum_k \mathscr{L}(j\Delta_1^{+k10^{-5}}, j\Delta_1 + (k+1)10^{-5}) - 10^{-5}F'(S_{k10^{-5}}X^{(0)})$$

$$\times \prod_{l>k} [E + 10^{-5}F'(S_{l10^{-5}}X^{(0)})]$$

We can estimate each term of the last expression by its norm. A rough estimation shows that

$$\|\mathscr{L}(j\Delta_1, j\Delta_1 + k10^{-5})\| \leqslant e^{200\Delta_1} = e^2 < 9$$

In the same way

$$\left\| \prod_{l_1<k} [E + 10^{-5}F'(S_{l_1 10^{-5}}X^{(0)})] \right\| < 9$$

The difference

$$\mathscr{L}(j\Delta_1 + k10^{-5}, j\Delta_1 + (k+1)10^{-5}) - [E + 10^{-5}F'(S_{j\Delta_1+k10^{-5}}X^{(0)})]$$

can be estimated as follows. Let us consider two systems of matrix equations

$$\frac{d\bar{X}}{dt} = F'(S_{j\Delta_1+k10^{-5}}X^{(0)})\bar{X}, \qquad \frac{dX}{dt} = F'(S_{j\Delta_1+k10^{-5}+t}X^{(0)})X$$

for $0 \leqslant t \leqslant 10^{-5}$ with the initial conditions $\bar{X} = X = E$. Let us put $Z = X - \bar{X}$. Then $Z(0) = 0$ and

$$\frac{dZ}{dt} = F'(S_{j\Delta_1+k10^{-5}+t}X^{(0)})Z + [F'(S_{t+j\Delta_1+k10^{-5}}X^{(0)}) - F'(S_{j\Delta_1+k10^{-5}}X^{(0)}]\bar{X}$$

In our case

$$\|F'(S_{t+j\Delta_1+k10^{-5}}X^{(0)}) - F(S_{j\Delta_1+k10^{-5}}X^{(0)})\| \leqslant 2 \times 10^{-3}$$

for $0 \leqslant t \leqslant 10^{-5}$, $\|F'(S_{j\Delta_1+k10^{-5}+t}X^{(0)})\| \leqslant 200$. With the help of continuous induction it is easy to show that $\|Z\| \leqslant 400 \times 10^{-10} = 4 \times 10^{-8}$. From the other side,

$$\|\bar{X} - [E + 10^{-5}F'(S_{j\Delta_1+k10^{-5}}X^{(0)})]\| \leqslant 10^{-8}$$

Finally we get

$$\left\|\mathscr{L}(j\Delta_1, (j+1)\Delta_1) - \prod_{k=0}^{999} [E + 10^{-5}F'(S_{j\Delta_1+k10^{-5}}X^{(0)})]\right\| \leqslant 6 \times 10^{-5}$$

From the inductive hypothesis we know the coefficient d_j and therefore $\|\mathscr{L}(t_1, t_2)\| \leqslant 2(\tilde{C}_1 + d_j)$ for arbitrary t_1, t_2 such that $0 \leqslant t_1 \leqslant t_2 \leqslant j\Delta_1$. This permits us to apply the results of Section 5 and to estimate the error $\|S_{k\Delta}X^{(0)} - X_k\|$. In view of (13) and (14),

$$\|S_{k\Delta}X^{(0)} - X_k\| \leqslant 2(\tilde{C}_1 + d_j)k \times 10^{-7}A$$

where the value A is found from (13) where C_1 is replaced by $2(\tilde{C}_1 + d_j)$. Now we can write

$$\|F'(S_{k\Delta}X^{(0)}) - F'(X_k)\| \leqslant 200\|S_{k\Delta}X^{(0)} - X_k\|$$

and get the estimate of the second term in (20). Collecting together all the estimates, we obtain the estimate of C_1. In our case it turns out that $C_1 = 25$. Now we can estimate the value of K_0. Substitution of all the constants in (8) gives the inequality $K_0 \leqslant 5 \times 10^5$.

The estimation of ϵ was described in Section 2. In our case in formula (2), $\|\bar{X} - X^{(0)}\| \leqslant 4 \times 10^{-9}$. Estimating the norm $\|\bar{X} - X^{(1)}\|$ in the manner described in Section 2, we obtain $\epsilon \leqslant 10^{-8}$.

For the norm $A = \|(L - E)^{-1}\|$ we have the inequality $A \leqslant 21$. Let us put $\bar{\rho}_0 = \frac{1}{3} \times 10^{-6}$. Then in our case

$$\|(L - E)^{-1}\|(\epsilon/\bar{\rho}_0 + K_0\bar{\rho}_0) \leqslant 21(10^{-8} \times 3 \times 10^6 + 10^{-6} \times 10^4) < 1$$

Thus the main theorem is proved:

Main Theorem. In the Lorenz model with parameters $r = 28$, $\sigma = 6$, $b = 8/3$ a closed orbit exists. This orbit intersects the $\frac{1}{3} \times 10^{-6}$-neighborhood of the point (3.5007871847; 3.3303317847; 27).

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